Matroids I

Definition: Let E be a finite set and $\mathcal{F} \subseteq 2^E$. A set system $(E, \mathcal{F})$ is called a **matroid** if it satisfies

(M1) $\emptyset \in \mathcal{F}$;

(M2) If $X \subseteq Y \in \mathcal{F}$ then $X \in \mathcal{F}$;

(M3) If $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there is $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.

Elements in $\mathcal{F}$ are called **independent sets**.

1: Let E be any finite set and $k \geq 0$. Let $\mathcal{F} = \{F \subseteq E : |F| \leq k\}$. Show that $(E, \mathcal{F})$ is a matroid. *(It is called **uniform matroid**.)*

Solution: All three axioms are clearly satisfied.

2: Let E be the set of columns of a matrix A over some field.
Let $\mathcal{F} = \{F \subseteq E :$ the columns in F are linearly independent$\}$. Show that $(E, \mathcal{F})$ is a matroid.
*(It is called **linear** or **vector matroid**.)*

Solution: (M1) and (M2) are clearly satisfied. (M3) is well known from linear algebra.

3: Let E be the set of edges of an undirected graph G. Let $\mathcal{F} = \{F \subseteq E : (V(G), F)$ is a forest$\}$. Show that $(E, \mathcal{F})$ is a matroid. *(It is called **graphic** or **cycle matroid**.)*

Solution: (M1) and (M2) are clearly satisfied. For (M3), consider what is the number of connected components in X and Y. Since the one for Y is smaller, there must be and edge in X connecting two components in Y.

Maximal independent sets in $\mathcal{F}$ are called **bases** (see linear matroid).

Minimal dependent sets (means not independent) in $\mathcal{F}$ are called **circuits** or **cycles** (see graphic matroid).

Motivation for Matroids

Let $(E, \mathcal{F})$ be a matroid. Let $c : E \to \mathbb{R}_+$. Find $X \in \mathcal{F}$ such that $\sum_{e \in X} c(e)$ is maximized.

Notice that this would be the same as *maximum cost spanning tree* for graphic matroid.

$(E, \mathcal{F})$ being a matroid $\Rightarrow$ greedy algorithm works

1. Sort E such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_m)$
2. Let $F = \emptyset$
3. for i in 1 to m
4.   if $\{e_i\} \cup F \in \mathcal{F}$ then $F := F \cup \{e_i\}$.

4: Show that the greedy algorithm is correct. (Hint: similar to Kruskal's algorithm - consider optimal $F^\star$ and investigate the difference of F and $F^\star$.)

Solution: Let F be $a_1, a_2, \ldots, a_m$ and $F^\star$ be $b_1, b_2, \ldots, b_n$. Assume $c(a_1) \geq c(a_2) \geq \cdots$ and $c(b_1) \geq c(b_2) \geq \cdots$. Let k be the smallest such that $c(a_k) < c(b_k)$ (if there is no such k, F is also optimal). Consider $A = \{a_1, \ldots, a_{k-1}\}$ and $B = \{b_1, \ldots, b_k\}$. Application of (M3) on A and B gives $b \in B$ such that $A \cup \{b\}$ is independent, this contradicts the choice of a_k in the greedy algorithm instead of choosing b.

5: Let $(E, \mathcal{F})$ satisfy (M1) and (M2). Suppose that the greedy algorithm works for all $c : E \to \mathbb{R}_+$. Show that $(E, \mathcal{F})$ is a matroid.

Solution: Let $X, Y \in \mathcal{F}$ and $|X| > |Y|$. Assign cost

$$c(e) = \begin{cases} 1+\varepsilon & \text{if } e \in Y \\ 1 & \text{if } e \in X \setminus Y \\ 0 & \text{otherwise} \end{cases}$$

Where $0 < \varepsilon < |Y|$. The greedy algorithm picks all elements from Y. But X is independent and $c(X) > c(Y)$. Since the greedy algorithm works, it must also pick some $x \in X \setminus Y$.

6: Show that all bases of a matroid have the same cardinality.

Solution: $\Rightarrow$ If bases B_1 and B_2 have different cardinality, then $(M3)$ gives contradiction with maximality of the smaller one.

Theorem 13.9 Let E be a finite set and $\mathcal{B} \subseteq 2^E$. $\mathcal{B}$ is the set of bases of some matroid iff $\mathcal{B}$ satisfies

(B1) $\mathcal{B} \neq \emptyset$;

(B2) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \setminus B_2$ there exists $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

7: Show that matroids satisfy (B1) and (B2).

Solution: Easy consequence of (M2) and (M3).

8: Show that if $\mathcal{B}$ satisfies (B1) and (B2), then all bases in $\mathcal{B}$ have the same cardinality.

Solution: Let $|B_1| > |B_2|$ and $|B_1 \cap B_2|$ is maximized. Use of (B2) gives a contradiction with maximality of $|B_1 \cap B_2|$.

9: For $\mathcal{B}$ satisfying (B1) and (B2) find a matroid $(E, \mathcal{F})$.

Solution: Let

$$\mathcal{F} = \{F \subseteq E : F \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

By (B1), (M1) and (M2) are satisfied. Verify (M3). Let $X, Y \in F$ with $|X| > |Y|$. If $Y \subseteq X$, then any element of X will do. Otherwise suppose $X \in B_1$ and $Y \in B_2$, where $|B_1 \cap B_2|$ is maximized. If $B_2 \cap (X \setminus Y) \neq \emptyset$, then Y is easy to extend.
Assume $B_2 \cap (X \setminus Y) = \emptyset$. Then

$$|B_1 \cap B_2| + |Y \setminus B_1| + |(B_2 \setminus B_1) \setminus Y| = |B_2| = |B_1| \geq |B_1 \cap B_2| + |X \setminus Y|$$

Observe $|X \setminus Y| > |Y \setminus X| \geq |Y \setminus B_1|$. Hence $(B_2 \setminus B_1) \setminus Y \neq \emptyset$. Pick $x \in (B_2 \setminus B_1) \setminus Y$ and use (B2) to get a contradiction with $|B_1 \cap B_2|$ being maximized.